Eur. Phys. J. B **5**, 555–557 (1998)

THE EUROPEAN PHYSICAL JOURNAL B

The new identity for the scattering matrix of exactly solvable models*

V. Korepin[1,a] and N. Slavnov[2]

[1] ITP, SUNY at Stony Brook, NY 11794-3840, USA
[2] Steklov Mathematical Institute, Gubkina 8, Moscow 117966, Russia

Received: 4 December 1997 / Accepted: 17 March 1998

Abstract. We discovered a simple quadratic equation, which relates scattering phases of particles on Fermi surface. We consider one-dimensional Bose gas and XXZ Heisenberg quantum spin chain.

PACS. 11.55.Ds Exact S matrices – 05.30.-d Quantum statistical mechanics – 05.50.+q Lattice theory and statistics; Ising problems

1 Introduction

In order to define everything precisely we consider a specific model. Let us concentrate our attention on Bose gas with delta interaction (quantum Nonlinear Schrödinger equation).

The Hamiltonian of the model is

$$H = \int dx \left(\partial_x \Psi^\dagger(x)\partial_x\Psi(x) + c\Psi^\dagger(x)\Psi^\dagger(x)\Psi(x)\Psi(x) - h\Psi^\dagger(x)\Psi(x)\right). \quad (1.1)$$

Here $0 < c < \infty$, $h > 0$ are a coupling constant and a chemical potential respectively. The canonical Bose fields $\Psi(x,t)$, $\Psi^\dagger(x,t)$, $(x, t \in \mathbf{R})$ obey the standard commutation relation

$$[\Psi(x,t), \Psi^\dagger(y,t)] = \delta(x-y). \quad (1.2)$$

They act in the Fock space with a vacuum vector $|0\rangle$:

$$\Psi(x,t)|0\rangle = 0. \quad (1.3)$$

Alternatively the model can be formulated on the language of many-body quantum mechanics. In this case the Hamiltonian of the system of N identical Bose particles can represented as

$$\mathcal{H}_N = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{N\ge j>k\ge 1} \delta(x_j - x_k) - hN. \quad (1.4)$$

In this form the model is called Bose gas with delta interaction. For nonzero value of the coupling constant the Pauli principal is valid (chapter VII of [1]).

The model was solved by Bethe Ansatz [2]. The ground state is a Fermi sphere. In order to describe it precisely, it is convenient to introduce spectral parameter λ (similar to rapidity). The derivative of the momentum of the particle with respect to the spectral parameter is

$$\frac{\partial k(\lambda)}{\partial \lambda} = 2\pi\rho(\lambda), \quad (1.5)$$

where the function $\rho(\lambda)$ is defined by an integral equation

$$\rho(\lambda) - \frac{1}{2\pi}\int_{-q}^{q} K(\lambda,\mu)\rho(\mu)\,d\mu = \frac{1}{2\pi}\cdot \quad (1.6)$$

Here q is the value of the spectral parameter on the Fermi surface, and the kernel of an integral operator is

$$K(\lambda,\mu) = \frac{2c}{c^2 + (\lambda-\mu)^2}\cdot \quad (1.7)$$

One can prove that the integral operator $\hat{I} - \frac{1}{2\pi}\hat{K}$ is not degenerated, and hence, the equation (1.6) has unique solution ([3], chapter I of [1]). The density of the gas is given by

$$D = \int_{-q}^{q} \rho(\lambda)\,d\lambda. \quad (1.8)$$

There is one type of particles in the model. It is defined at $\lambda \ge q$ or $\lambda \le -q$. The energy of the particle $\varepsilon(\lambda)$ is

$$\varepsilon(\lambda) - \frac{1}{2\pi}\int_{-q}^{q} K(\lambda,\mu)\varepsilon(\mu)\,d\mu = \lambda^2 - h. \quad (1.9)$$

It vanishes on the Fermi surface $\varepsilon(\pm q) = 0$. The momentum is

$$k(\lambda) = \lambda + \int_{-q}^{q} \theta(\lambda-\mu)\rho(\mu)\,d\mu. \quad (1.10)$$

---

* Dedicated to J. Zittartz on the occasion of his 60th birthday

[a] e-mail: korepin@insti.physics.sunysb.edu

Here

$$\theta(\lambda) = i \ln\left(\frac{ic+\lambda}{ic-\lambda}\right). \tag{1.11}$$

One can calculate a scattering matrix of particle with spectral parameter λ on another particle with spectral parameter μ (chapter I of [1]). There is no multi-particle production or reflection.The transition coefficient is equal to

$$\exp\{2\pi i F(\lambda|\mu)\}. \tag{1.12}$$

The phase $F(\lambda|\mu)$ is defined by an integral equation

$$F(\lambda|\mu) - \frac{1}{2\pi}\int_{-q}^{q} K(\lambda,\nu)F(\nu|\mu)\,d\nu = \frac{1}{2\pi}\theta(\lambda-\mu). \tag{1.13}$$

The most important are scattering phases of particles on the Fermi edges $F(q|q)$ and $F(q|-q)$. In this paper we shall prove the identity

$$\det\begin{pmatrix} 1-F(q|q) & F(q|-q) \\ -F(-q|q) & 1+F(-q|-q)\end{pmatrix} = 1. \tag{1.14}$$

This is the main result of the paper. Another way to rewrite this identity is

$$\Big(1-F(q|q)\Big)^2 - F^2(q|-q) = 1. \tag{1.15}$$

Here we have used the property $F(-\lambda|-\mu) = -F(\lambda|\mu)$, which follows from the antisymmetry of $\theta(\lambda-\mu) = -\theta(\mu-\lambda)$.

This identity also permit us to relate "fractional" charge to the phase shift on the Fermi surface. Fractional charge $\mathcal{Z}$ appears in the formulæ for finite size corrections (chapter I of [1]). This value is necessary for conformal description of the model (chapter XVIII of [1]) and it is equal to

$$\mathcal{Z} = 2\pi\rho(q). \tag{1.16}$$

Using the equations (1.6, 1.13) for $\rho(\lambda)$ and $F(\lambda|\mu)$, one can find the relationship between the fractional charge and the scattering phase on the Fermi surface

$$\mathcal{Z} = 1 + F(q|-q) - F(q|q). \tag{1.17}$$

Indeed, it follows from (1.6) that

$$[2\pi\rho(\lambda)-1] - \frac{1}{2\pi}\int_{-q}^{q} K(\lambda,\mu)[2\pi\rho(\mu)-1]\,d\mu = \frac{1}{2\pi}[\theta(\lambda+q)-\theta(\lambda-q)]. \tag{1.18}$$

Comparing this equation with (1.13) we find

$$2\pi\rho(\lambda) = 1 + F(\lambda|-q) - F(\lambda|q), \tag{1.19}$$

what, in turns, implies (1.17). The identity (1.15) allows us to find new relation

$$\mathcal{Z}^{-1} = 1 - F(q|-q) - F(q|q). \tag{1.20}$$

This identity helps to prove a consistency between conformal description of temperature correlation functions at low temperatures and asymptotic formulas for time and temperature dependent correlation functions, arising from determinant representation of correlation functions [5].

2 The proof of the main identity

In this section we give the proof of the identity (1.14). In order to do this, one should calculate the derivatives of the function $F(\lambda|\mu)$ with respect to λ, μ and q. Using the basic equation (1.13), we have

$$\frac{\partial F(\lambda|\mu)}{\partial\lambda} - \frac{1}{2\pi}\int_{-q}^{q} K(\lambda,\nu)\frac{\partial F(\nu|\mu)}{\partial\nu}\,d\nu = \frac{1}{2\pi}K(\lambda,\mu) - \frac{1}{2\pi}K(\lambda,q)F(q|\mu) + \frac{1}{2\pi}K(\lambda,-q)F(-q|\mu), \tag{2.1}$$

$$\frac{\partial F(\lambda|\mu)}{\partial\mu} - \frac{1}{2\pi}\int_{-q}^{q} K(\lambda,\nu)\frac{\partial F(\nu|\mu)}{\partial\mu}\,d\nu = -\frac{1}{2\pi}K(\lambda,\mu), \tag{2.2}$$

$$\frac{\partial F(\lambda|\mu)}{\partial q} - \frac{1}{2\pi}\int_{-q}^{q} K(\lambda,\nu)\frac{\partial F(\nu|\mu)}{\partial q}\,d\nu = \frac{1}{2\pi}K(\lambda,q)F(q|\mu) + \frac{1}{2\pi}K(\lambda,-q)F(-q|\mu). \tag{2.3}$$

Here we have used that $\frac{\partial}{\partial\lambda}\theta(\lambda-\mu) = K(\lambda,\mu)$.

As we have mentioned already, the re-solvent of the operator $\hat{I} - \frac{1}{2\pi}\hat{K}$ exists and it is equal to

$$\hat{R} = \left(\hat{I} - \frac{1}{2\pi}\hat{K}\right)^{-1}\frac{1}{2\pi}\hat{K}. \tag{2.4}$$

The derivatives of the function $F(\lambda|\mu)$ can be expressed in terms of the re-solvent

$$\begin{aligned}\frac{\partial F(\lambda|\mu)}{\partial\lambda} &= R(\lambda,\mu) - R(\lambda,q)F(q|\mu) + R(\lambda,-q)F(-q|\mu)\\ \frac{\partial F(\lambda|\mu)}{\partial\mu} &= -R(\lambda,\mu)\\ \frac{\partial F(\lambda|\mu)}{\partial q} &= R(\lambda,q)F(q|\mu) + R(\lambda,-q)F(-q|\mu).\end{aligned} \tag{2.5}$$

Using these equations one can find the complete derivatives with respect to q of functions $F(q|q)$, $F(q|-q)$ *etc.*:

$$\frac{d}{dq}F(q|\pm q) = \left(\frac{\partial}{\partial\lambda} \pm \frac{\partial}{\partial\mu} + \frac{\partial}{\partial q}\right) F(\lambda|\mu)\Big|_{\substack{\lambda = q\\ \mu=\pm q}}, \quad \frac{d}{dq}F(-q|\pm q) = \left(-\frac{\partial}{\partial\lambda} \pm \frac{\partial}{\partial\mu} + \frac{\partial}{\partial q}\right) F(\lambda|\mu)\Big|_{\substack{\lambda = -q\\ \mu=\pm q}}. \tag{2.6}$$

Substituting here equations (2.5) we find

$$\begin{aligned} &\frac{d}{dq}F(q|-q) = 2R(q,-q)\Big(1+F(-q|-q)\Big),\\ &\frac{d}{dq}F(-q|q) = -2R(-q,q)\Big(1-F(q|q)\Big),\\ &\frac{d}{dq}F(q|q) = 2R(q,-q)F(-q|q),\\ &\frac{d}{dq}F(-q|-q) = 2R(-q,q)F(q|-q). \end{aligned} \tag{2.7}$$

Now it is sufficient to take the derivative with respect to q of the l.h.s. of the equation (1.14):

$$\begin{aligned} &\frac{d}{dq}\det\begin{pmatrix}1-F(q|q) & F(q|-q)\\ -F(-q|q) & 1+F(-q|-q)\end{pmatrix}\\ &= -\frac{dF(q|q)}{dq}\Big(1+F(-q|-q)\Big) + \Big(1-F(q|q)\Big)\frac{dF(-q|-q)}{dq}\\ &\quad + \frac{dF(q|-q)}{dq}F(-q|q) + F(q|-q)\frac{dF(-q|q)}{dq}\\ &= -2R(q,-q)F(-q|q)\Big(1+F(-q|-q)\Big)\\ &\quad +2R(-q,q)F(q|-q)\Big(1-F(q|q)\Big)\\ &\quad +2R(q,-q)F(-q|q)\Big(1+F(-q|-q)\Big)\\ &\quad -2R(-q,q)F(q|-q)\Big(1-F(q|q)\Big). \end{aligned} \tag{2.8}$$

On the other hand, it is clear that for $q=0$ we have

$$\det\begin{pmatrix}1-F(q|q) & F(q|-q)\\ -F(-q|q) & 1+F(-q|-q)\end{pmatrix}\Big|_{q=0} = 1. \tag{2.9}$$

Thus, the identity (1.14) is proved. We would like to emphasize the we did not use the explicit expressions for the kernel $K(\lambda,\mu)$ and the function $\theta(\lambda-\mu)$. In fact, we have used only three properties:

a) the existence of the re-solvent of the operator $\hat{I} - \frac{1}{2\pi}\hat{K}$;

b) the kernel $K(\lambda,\mu)$ and the function $\theta(\lambda-\mu)$ depend on the difference;

c) the derivative of $\theta(\lambda-\mu)$ is equal to the kernel $K(\lambda,\mu)$.

In order to reduce (1.14) to the identity (1.15) one should use also the antisymmetry property $\theta(\lambda-\mu) = -\theta(\mu-\lambda)$.

Thus, the quadratic identity for the scattering phase is valid for a wide class of completely integrable models, but not only for the one-dimensional Bose gas. In particular, it is valid for scattering phases of elementary particles (spin waves) of XXZ Heisenberg spin chain in a magnetic field ([4], see also chapter II of [1]).

This work was supported by the National Science Foundation (NSF) Grant No. PHY-9321165, and the Russian Foundation of Basic Research Grant No. 96-01-00344.

References

1. V.E. Korepin, N.M. Bogoliubov, A.G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge University Press, 1993).
2. E.H. Lieb, W. Liniger, Phys. Rev. **130**, 1605 (1963).
3. C.N. Yang, C.P. Yang, J. Math. Phys. **10**, 1115 (1969).
4. C.N. Yang, C.P. Yang, Phys. Rev. **150**, 321 (1966).
5. V.E. Korepin, N.A. Slavnov, Phys. Lett. A **236**, 201 (1997).